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BRST Quantization of a Scalar Particle in a Curved Background

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The BRST formalism is employed to quantize a scalar particle and interactions with an external scalar field $\Phi(x^\mu)$ and vector gauge field $A_\mu(x^\mu)$ in the background of an arbitrary gravitational field. The second-quantized actions are obtained.

1. INTRODUCTION

The principle of local gauge invariance is considered to be the cause of all the fundamental forces of nature. Consequently, the underlying Lagrangians are singular. The BRST approach (Becchi *et al.*, 1976; Tyutin, 1975) offers an elegant and powerful technique to deal with the quantization of systems possessing local symmetries. The reparametrization invariance for this special class of singular Lagrangians needs attention when we have to fix the gauge. The actions for a relativistic particle, for the gravitational field and for a superstring belong to this class. Recently, considerable attention has been focused on the BRST quantization of a relativistic particle following the work of Siegel (1984, 1985), who demonstrated how the quantization of a free particle provides a clue to covariant quantization of the string field theory.

The purpose of this paper is to treat the evolution of a spinless particle in a curved background metric and quantize this system in the BRST Hamiltonian formalism.

2. FREE PARTICLE

The action is (Weinberg, 1972)

$$L=\int ds\,\mathscr{L}=-m\int ds\left[-g_{\mu\nu}(x^\mu(s))\,\frac{dx^\mu(s)}{ds}\,\frac{dx^\nu(s)}{ds}\right]^{1/2} \tag{2.1}$$

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Here $g_{\mu\nu}(x^\mu(s))$ is taken as a background gravitational field and the space-time coordinate x^μ of the particle is parametrized by s. The action (2.1) is invariant under the reparametrization $s \to f(s)$, f being an arbitrary monotonically increasing function of s, and it leaves the endpoints of the integration unchanged.

We adopt Siegel's strategy in order to quantize the theory described by (2.1) without applying a standard variational principle.

In order to apply the BRST Hamiltonian formalism, we define

$$p_\mu \equiv \frac{\partial \mathscr{L}}{\partial \dot{x}^\mu} = m g_{\mu\nu}(x^\mu)(-g_{\lambda\rho}\dot{x}^\lambda \dot{x}^\rho)^{-1/2} \tag{2.2}$$

with $\dot{x}^\mu(s) = dx^\mu(s)/ds = \partial x^\mu$. It follows from (2.2) that

$$p_\mu p_\nu g^{\mu\nu}(x^\mu) + m^2 = 0 \tag{2.3}$$

which is the analog of the mass-shell condition for a free particle. We introduce a Lagrange multiplier $e(s)$ to enforce the constraint (2.3). The Hamiltonian action takes the form

$$L_0 = \int ds\{p_\mu \dot{x}^\mu - e(s)[p_\mu p_\nu g^{\mu\nu}(x^\mu) + m^2]\} \tag{2.4}$$

where p_μ, x^μ and e are functions of s, and e is s-metric, which preserves s-reparametrization invariance

$$\delta x^\mu = \xi \dot{x}^\mu, \qquad \delta p = \xi \dot{p}^\mu, \qquad \delta e = \partial(\xi e), \qquad \delta g^{\mu\nu} = \xi \dot{g}^{\mu\nu} \tag{2.5}$$

The variation of L_0 is a total derivative

$$L_0 = \int ds\, \partial(\xi\{p_\mu \dot{x}^\mu - e(s)[p_\mu p_\nu g^{\mu\nu} + m^2]\}) \tag{2.6}$$

BRST transformations are now defined as in Yang–Mills theory, with $\xi \to i\varepsilon\theta$, where ε is a constant anticommuting parameter, and θ is the ghost, $\hat{\theta}$ is the antighost; and B is the Nakanishi–Lautrup auxiliary field (Nakanishi, 1966; Lautrup, 1967). The Lagrangian and transformations become

$$L = L_0 + L_{gf} + L_{fp} = \int [p_\mu \dot{x}^\mu - e(p_\mu p_\nu g^{\mu\nu} + m^2) + B(e-1) - i\hat{\theta}(\partial\theta)]\, ds$$

$$\delta x^\mu = i\varepsilon\theta\, \partial x^\mu, \qquad \delta p^\mu = i\varepsilon\theta\, \partial p^\mu, \qquad \delta e = i\varepsilon\, \partial(\theta e), \tag{2.7}$$

$$\delta\theta = i\varepsilon\theta\, \partial\theta, \qquad \delta\hat{\theta} = eB, \qquad \delta B = 0, \qquad \delta g^{\mu\nu} = i\varepsilon\theta\, \partial g^{\mu\nu}$$

We redefine B

$$B = \hat{B} + (p_\mu p_\nu g^{\mu\nu} + m^2) - i(\partial\hat{\theta})\theta \tag{2.8}$$

The Lagrangian and transformations become

$$L=\int [p_\mu \dot{x}^\mu-(p_\mu p_\nu g^{\mu\nu}+m^2)-i\hat{\theta}\,\partial\theta+\hat{B}(e-1)]\,ds$$

$$\delta\hat{\theta}=\varepsilon[\hat{B}+(p_\mu p_\nu g^{\mu\nu}+m^2)-i(\partial\hat{\theta})\theta] \tag{2.9}$$

$$\delta\hat{B}=i\varepsilon\theta\,\partial\hat{B}, \qquad \text{rest as before}$$

Now, following Siegel (1984, 1985), we can consistently set $\hat{B}=0$ and still maintain nilpotency off shell, giving the final result

$$L=\int [p_\mu\,\partial x^\mu-(p_\mu p_\nu g^{\mu\nu}+m^2)-i\hat{\theta}\,\partial\theta]\,ds$$

$$\delta x^\mu=i\varepsilon\theta\,\partial x^\mu, \qquad \delta p^\mu=i\varepsilon\theta\,\partial p^\mu, \qquad \delta\theta=i\varepsilon\theta\,\partial\theta, \tag{2.10}$$

$$\delta\hat{\theta}=i\theta\,\partial\hat{\theta}+\varepsilon(p_\mu p_\nu g^{\mu\nu}+m^2), \qquad \delta g^{\mu\nu}=i\varepsilon\theta\,\partial g^{\mu\nu}$$

The variation of L is a total derivative

$$\delta L=\int ds\,\partial(i\varepsilon\theta L) \tag{2.11}$$

The BRST charge is

$$Q=\theta(p_\mu p_\nu g^{\mu\nu}+m^2) \tag{2.12}$$

Now we present the second-quantized action for the system. Let us consider a real scalar field $\phi(x,\theta)$ having the following expansion in θ:

$$\phi(x,\theta)=\phi(x)+i\theta\psi(x) \tag{2.13}$$

and under the BRST transformation, $\delta\phi=i\varepsilon Q\phi$. The BRST charge (2.12) takes the form (we replace $p_\mu\to i\partial/\partial x^\mu$ in the Schrödinger picture)

$$Q=\theta(-\Box+m^2) \tag{2.14}$$

where

$$\Box=-(-g)^{-1/2}\frac{\partial}{\partial x^\mu}(-g)^{1/2}g^{\mu\nu}\frac{\partial}{\partial x^\nu} \tag{2.15}$$

is the d'Alembertian in a curved manifold.

Now, following Siegel (1984, 1985), we introduce an operator D satisfying

$$[D,Q]=0 \tag{2.16}$$

The second-quantized action

$$L=-\tfrac{1}{2}\int \sqrt{-g}\,d^4x\,d\theta\,\phi(x,\theta)\,D\phi(x,\theta) \tag{2.17}$$

is BRST invariant due to (2.16) and nilpotency of Q. We choose

$$D \equiv [\theta\, \partial/\partial\theta, 0] = \theta(-\Box + m^2) \tag{2.18}$$

Inserting equations (2.13) and (2.18) in (2.17) and using $\int d\theta\, \theta = 1$ and $\theta^2 = 0$, we obtain

$$L = \tfrac{1}{2} \int \sqrt{-g}\, dx^4\, \phi(x)(-\Box + m^2)\phi(x) \tag{2.19}$$

This is the action for propagation of a scalar field in an arbitrary background gravitational field. The above action differs from the standard form of the Lagrangian

$$\tfrac{1}{2}(-g)^{1/2}(g^{\mu\nu}\, \partial_\mu\phi\, \partial_\mu\phi - m^2) \tag{2.20}$$

3. INTERACTION

Interaction can be introduced by adding a $-e\Phi(x)$ term to (2.4), where $\Phi(x)$ is an external scalar field. By a method similar to Section 2, we give the Lagrangian of interaction

$$\mathscr{L} = p_\mu \dot{x}^\mu - (p_\mu p_\nu g^{\mu\nu} + \Phi(x) - i\hat{\theta}\, \partial\theta) \tag{3.1}$$

The BRST charge is $Q = \theta(-\Box + m^2 + \Phi)$.

The second-quantized action is

$$L = \tfrac{1}{2} \int \sqrt{-g}\, d^4x\, \phi(x)(-\Box + m^2 + \Phi)\phi(x) \tag{3.2}$$

The Φ may be interpreted as a self-interaction by identifying Φ with ϕ.

Interaction can be introduced by adding $-A_\mu(x^\mu)\, \partial x$, where $A_\mu(x^\mu)$ is an external vector gauge field. We have the Lagrangian

$$\mathscr{L} = [p_\mu \dot{x}^\mu - (p_\mu + A_\mu)(p_\nu + A_\nu) g^{\mu\nu} + m^2] - i\hat{\theta}\, \partial\theta \tag{3.3}$$

The BRST charge is

$$Q = \left[-\frac{1}{\sqrt{-g}} \left(\frac{\partial}{\partial x^\mu} - iA_\mu \right) \sqrt{-g}\, g^{\mu\nu} \left(\frac{\partial}{\partial x^\nu} - iA_\nu \right) + m^2 \right]$$

The second-quantized action is

$$L = \tfrac{1}{2} \int \sqrt{-g}\, dx^4\, \phi \left[-\frac{1}{\sqrt{-g}} \left(\frac{\partial}{\partial x^\mu} - iA_\mu \right) \sqrt{-g}\, g^{\mu\nu} \left(\frac{\partial}{\partial x^\nu} - iA_\nu \right) + m^2 \right] \phi \tag{3.4}$$

4. CONCLUSION

We have applied the BRST formulation to quantize a spinless particle and interactions with an external scalar field $\Phi(x)$ and vector gauge field $A_\mu(x)$ in the background of an arbitrary metric $g^{\mu\nu}(x)$. The Lagrangian, the gauge-fixed Lagrangian action, including the ghosts, and second-quantized actions are obtained for a free particle and interactions in a curved manifold following the prescriptions due to Siegel. Interaction can be introduced by changing $m^2 \to m^2 + \Phi(x)$ and $\partial/\partial x^\mu \to \partial/\partial x^\mu \to iA_\mu(x)$.

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